Exact Solutions to Einstein's Equations in Astrophysics

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Exact Solutions to Einstein's Field Equations (EFE)

- No unique or simple definition (Kramer et al. 1980; Stephani et al. 2003).
- EFE with a cosmological constant: $$G_{ab} + \Lambda g_{ab} = \kappa T_{ab}$$
- $G_{ab} \equiv R_{ab} - \frac{1}{2} R g_{ab}$: Einstein tensor representing the spacetime curvature; $\kappa = 8\pi G/c^4$
- g_{ab}: metric tensor, $R_{ab} \equiv R^{c}{}_{acb}$: Ricci tensor, R_{abcd} : Riemann tensor
- T_{ab} : the energy-momentum tensor representing the source of the gravitational field
- The EFE are 10 nonlinear partial differential equations in 4 independent variables
- Solutions are usually obtained by assuming symmetries on the metric and other simplifying restrictions

Given a source energy-momentum tensor, an exact solution generally means a solution to the Einstein equations where the spacetime metric functions are expressed in terms of elementary or well-known special functions.

Studying Exact Solutions to Einstein's Equations

- In the first edition of "Exact Solutions of Einstein's Field Equations" by Kramer, Stephani, Herlt, MacCallum and Schmutzer, Cambridge University Press, 1980, the authors collected 2000 papers on exact solutions.

- 4000 more papers collected in 1999 leading to the second edition in 2003, but a large fraction are re-derivations.

- There is the pure mathematical problem of deriving an exact solution and then the physical interpretation of the exact solution

- Interpreting the currently available solutions is commonly more encouraged in the field than deriving new ones

- Solutions have been invariantly classified using algebraic symmetry schemes such as Petrov or Segre types, and also groups of motions.

Petrov Algebraic Classifications of exact solutions

- Using the Weyl tensor (Petrov, Uch. Zapiski Kasan Gos. Univ. 1954; Pirani, 1957; Geheniau, 1957, Debever, 1959; Bel, 1959; Penrose, 1960)
- Weyl tensor for a spacetime with n dimensions is given by

$$C_{abcd} = R_{abcd} - \frac{2}{n-2}\left(g_{a[c}R_{d]b} - g_{b[c}R_{d]a}\right) + \frac{2}{(n-1)(n-2)} R\, g_{a[c}g_{d]b}$$

- For the classification, Weyl tensor is considered as an operator acting on the space of bivectors X^{ab}

$$\frac{1}{2} C^{ab}{}_{cd} X^{cd} = \lambda X^{ab}$$

- Symmetries of a spacetime are associated with the multiplicity of its eigenbivectors
- The classification also offers some insights on some physical properties

Petrov Type	Eigenbivectors/ Principal null directions
I	Four simple directions (most general)
II	One double and two simple
D	Two double null directions
III	One triple and one simple
N	One quadruple
O	The Weyl tensor vanishes

Segre Algebraic Classifications of exact solutions

Using the multiplicity of the eigenvectors of the Ricci tensor

$$R^a{}_b v^b = \lambda v^a$$

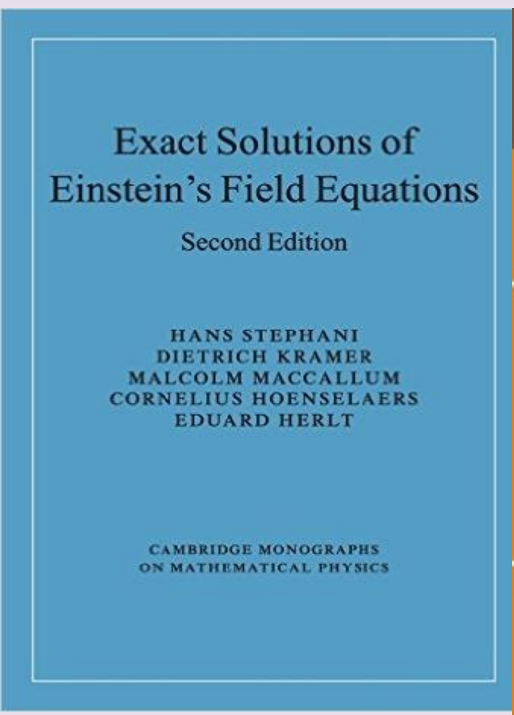
(Segre, Mem. R. Acad. Lincei 3a, 127, 1874, Plebanski, Acta Phys. Polo. 26, 963,1964)

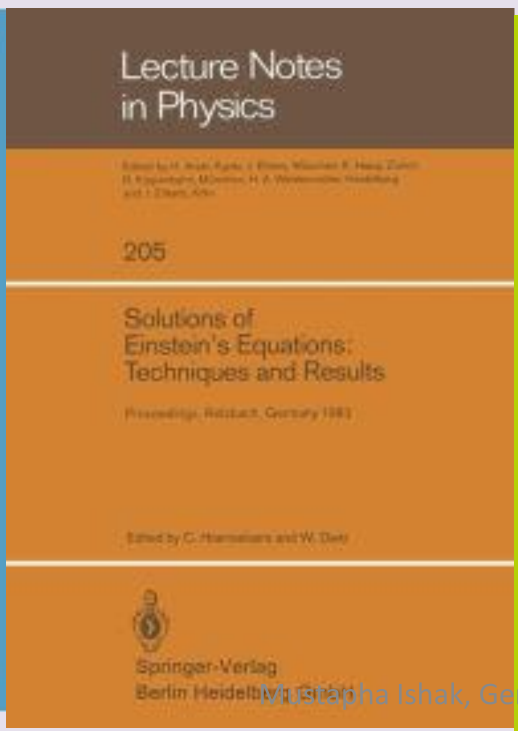
	Segre notation	Plebański notation	Physical interpretations
A1	$[111,1]$	$[S_1 - S_2 - S_3 - T]_{(1111)}$	
	$[11(1,1)]$	$[S_1 - S_2 - 2T]_{(111)}$	
	$[(11)1,1]$	$[2S_1 - S_2 - T]_{(111)}$	
	$[(11)(1,1)]$	$[2S - 2T]_{(11)}$	Non-null Maxwell field
	$[1(11,1)]$	$[S - 3T]_{(11)}$	
	$[(111),1]$	$[3S - T]_{(11)}$	Perfect fluid
	$[(111,1)]$	$[4T]_{(1)}$	Λ-term
A2	$[11,Z\overline{Z}]$	$[S_1 - S_2 - Z - \overline{Z}]_{(1111)}$	
	$[(11),Z\overline{Z}]$	$[2S - Z - \overline{Z}]_{(111)}$	
A3	$[11,2]$	$[S_1 - S - 2N]_{(112)}$	
	$[1(1,2)]$	$[S - 3N]_{(12)}$	
	$[(11),2]$	$[2S - 2N]_{(12)}$	
	$[(11,2)]$	$[4N]_{(2)}$	Null Maxwell field, pure rad.
B	$[1,3]$	$[S - 3N]_{(13)}$	
	$[(1,3)]$	$[4N]_{(3)}$	

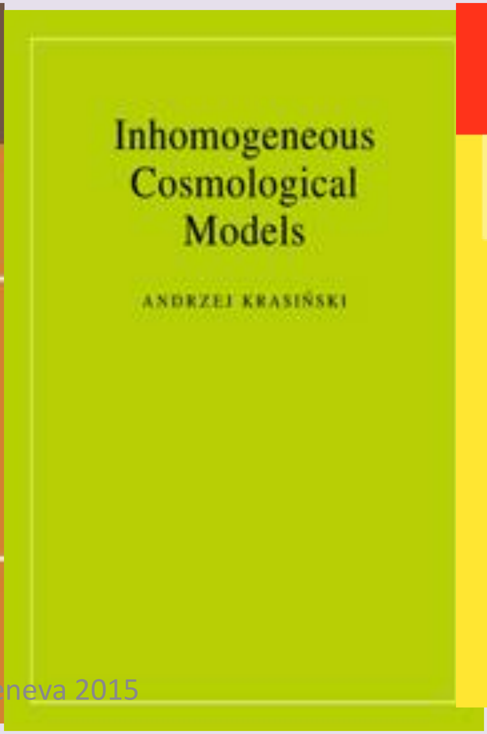
Classical books/catalogs of exact solutions

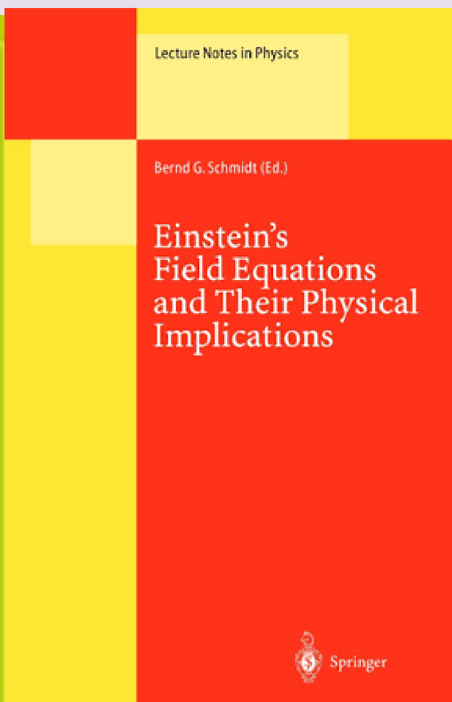
For example:

- Kramer, Stephani, Herlt, MacCallum and Schmutzer, Exact Solutions of Einstein's Field Equations, Cambridge University Press, 1980. (First edition)
- Stephani, Kramer, MacCallum, Hoenselaers, Herlt, Exact Solutions of Einstein's Field Equations Cambridge University Press, Cambridge, 2003. (Second edition)
- Krasinski, Inhomogeneous Cosmological Models, Cambridge University Press, Cambridge, 1997
- Ellis & van Elst, Cargèse Lectures 1998, Cosmological Models, gr-qc/981204









Online database of exact solutions (static)

http://www.personal.soton.ac.uk/rdi/database/ (J. Skea, 1997)

Faculty of Mathematical Studies
University of Southampton
United Kingdom

Symbolic Computation Group
Dept of Theoretical Physics
Instituto de Física
UERJ
Brazil

Welcome To The On-Line Invariant Classification Database.

Petrov Type

Petrov Type	I	II	III	D	N	O
Select	○	○	○	○	○	○

Segre Type

Segre Type	0	[1,(111)]	[(1,1)(11)]	[(211)]	[(1,1)11]	[1,1(11)]	[(1,11)1]
Notes	vacuum	perfect fluid	includes non-null e.-m.	radiation null fluid	boost isotropy	rotational isotropy	tachyon fluid
Select	○	○	○	○	○	○	○

Segre Type	[(21)1]	[2(11)]	[211]	[31]	[(31)]	[1,111]	[zz11]	[zz(11)]
Notes	null isotropy	[2(11)]	Plebanski Petrov II	Plebanski Petrov III	4 null directions coincide	general real	general complex	fails energy conditions
Select	○	○	○	○	○	○	○	○

Online database of exact solutions: grdb.org (non-static)

only the metric is stored, all calculations are done interactively using GRTensorII software (MI & Lake, CQG 19 (2002) 505-514)

Interactive Geometric Database

Including Exact Solutions of Einstein's Field Equations

Version 1.0

Initial development by

Mustapha Ishak and Kayll Lake

Public **Information** **Help** **Clear Results** **Public Input**

Private **Database** **List All** **Private Input**

Click to Search Database

Search by Keyword (e.g. Kruskal) Search by KSHMS (e.g. (13.24) p158)

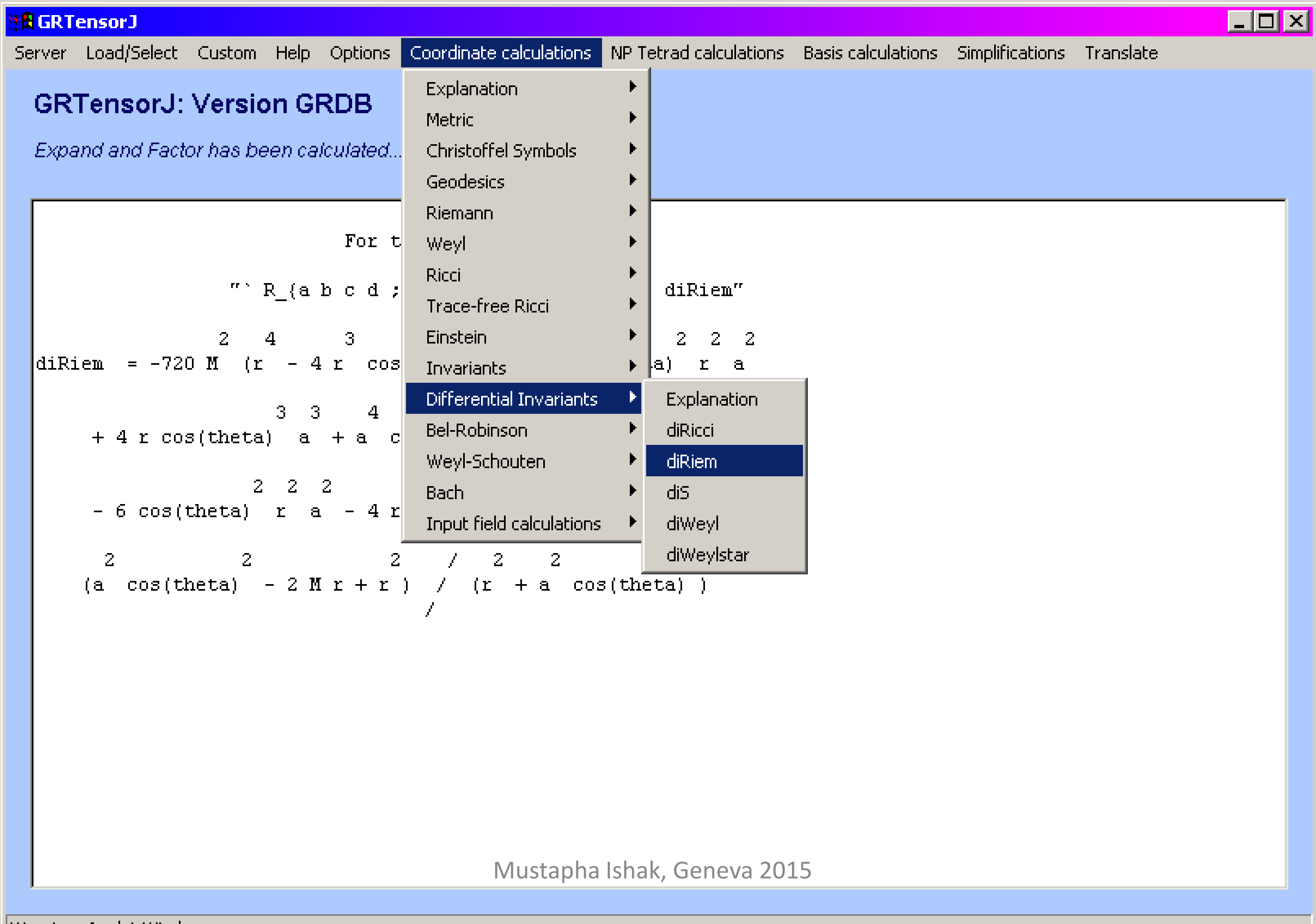
More search parameters in development

Click to Start Calculator

Search Results From Database

	Name	Description
1	kerrnewmannpuc	\`Contravariant NP tetrad for he Kerr Newman metric in Boyer Lindquist coordinates (u=a*cos(theta))\`
2	kerrma	\`The Kerr metric in Cartesian coordinates\`
3	kerrmb	\`The Kerr metric in Boyer Lindquist coordinates\`
4	kerrmc	\`The Kerr metric in Boyer Lindquist coordinates (u=a*cos(theta))\`
5	kerrmd	\`The Kerr metric in outgoing Eddington Finkelstein form (e.g. MTW Box 33.2)\`

The GRTensorJ calculator of grdb.org can upload a metric and perform tensor or tetrad calculations
(currently under further development!)



Applications of exact solutions in Astrophysics

- Slowly rotating stars and planets: The Schwarzschild solution
- Static black holes: The Schwarzschild solution
- Star interiors: e.g. Tolman, Buchdahl, Heintzmann solutions
- Neutron stars: Tolman VII and Durgapal solutions
- Rotating black holes: The Kerr-(Newman) solution
- Gravitational waves: gravitational plane wave exact solution
- Standard model of cosmology: The Friedmann-Lemaitre-Robertson-Walker solution
- Inhomogeneous Cosmological models: e.g. Lemaitre-Tolman-Bondi solutions, Szekeres solutions, Oleson solutions

Solar and planetary systems: The Schwarzschild exterior solution

(Schwarzschild, Sitz. Preuss. Akad. Wiss. 23, p189, Jan. 1916)

- Gravitational field in vacuum around a concentric spherical mass. In Schwarzschild coordinates:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \; d\varphi^2\right).$$

- Uniqueness theorems for the solution

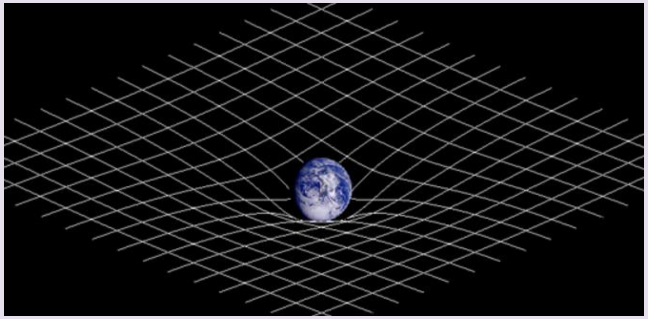
- A good description of the field around slowly rotating astrophysical objects such as planets, stars and static black-holes

- Expressed in other coordinates and Taylor expansions of its metric functions

- Played a role in planetary/solar tests of GR

- Played a role in gravitational lensing studies

- Used in construction of other exact solutions such as cosmological Swiss-Cheese models.



Credit: Johnstone using an image from NASA's Galileo spacecraft.

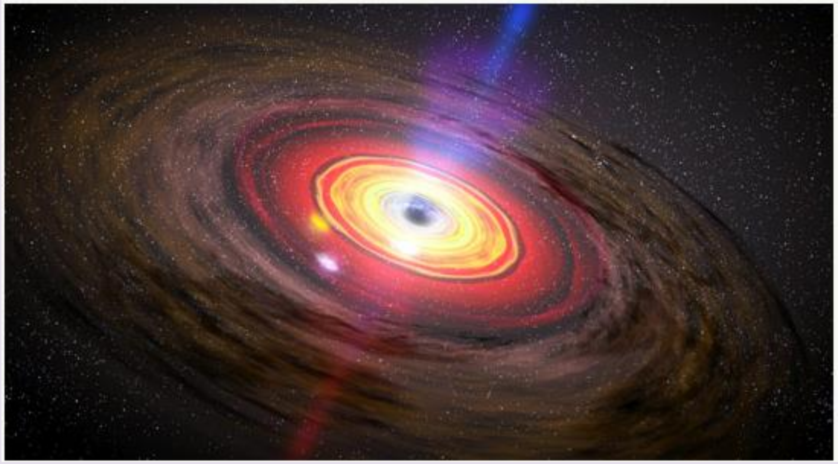
Rotating black holes: The Kerr exterior solution

(Kerr, Phys. Rev. Lett. 11 , 237–238, 1963)

- Describes the gravitational field around a rotating axially symmetric black hole
- Metric in Boyer-Lindquist coordinates (*J. Math. Phys.* **8** (2): 265, 1967) reads

$$\begin{aligned} ds^2 &= -\left[1 - \frac{2mr}{r^2 + a^2\cos^2\theta}\right] dt^2 - \frac{4mra\sin^2\theta}{r^2 + a^2\cos^2\theta}\, dt\, d\phi + \left[\frac{r^2 + a^2\cos^2\theta}{r^2 - 2mr + a^2}\right] dr^2 \\ &\quad + (r^2 + a^2\cos^2\theta)\, d\theta^2 + \left[r^2 + a^2 + \frac{2mra^2\sin^2\theta}{r^2 + a^2\cos^2\theta}\right] \sin^2\theta\, d\phi^2. \end{aligned}$$

- m the mass and **a** the angular momentum per unit mass.
- Uniqueness theorems for the solution
- Used to describe astrophysics of quasars
- Used to describe accreting flows in stellar-mass or super-massive black holes
- Used as starting point in some simulations of rotating black-holes



Credit: ESA/NASA

Application to compact star models: isolated Static Spherically Symmetric (SSS) solutions

- Over 130 distinct SSS solutions where studied in (Delgaty & Lake, Comp. Phys. Commun. 115, 395, 1998)
- Only 10% pass minimal physical acceptability conditions: e.g. positive definiteness of the pressure and density; subluminal adiabatic sound speed (Delgaty & Lake, Comp. Phys. Commun. 115, 395, 1998; MI, Chamandy, Neary, Lake, PRD, 64, 024005, 2001).
- The Schwarzschild interior solution (Sitz. Preuss. Akad. Wiss. 24, p424, Feb. 1916) does not pass the subluminal adiabatic sound speed test.

Some exact solutions that pass the tests:

- Buchdahl solution (Buchdahl, ApJ 147, 310 (1967))
- Finch and Skea solution (CQG. 6, 467 (1989))
  Heintzmann solution (Z. Phys. 228 489 (1969)).
- Durgapal solution (J. Phys. A 15 2637 (1982)). Lattimer & Prakash, Astrop. J. 550:426-442, 2001
- Tolman VII solution (Phys. Rev. 55, 364, 1939). Lattimer & Prakash, Phys. Rept. 442: 109-165,2007);

Neutron stars: the Tolman VII solution

- The metric in the original formulation by Tolman (Phys. Rev., 55, 364, 1939)

$$ds^2 = -B^2 \sin^2 \ln \sqrt{\frac{\sqrt{1-\frac{r^2}{R^2}+4\frac{r^4}{A^4}}+2\frac{r^2}{A^2}-\frac{1}{4}\frac{A^2}{R^2}}{C}}\, dt^2 + \left(1-\frac{r^2}{R^2}+4\frac{r^4}{A^4}\right)^{-1} dr^2 + r^2 d\Omega^2$$

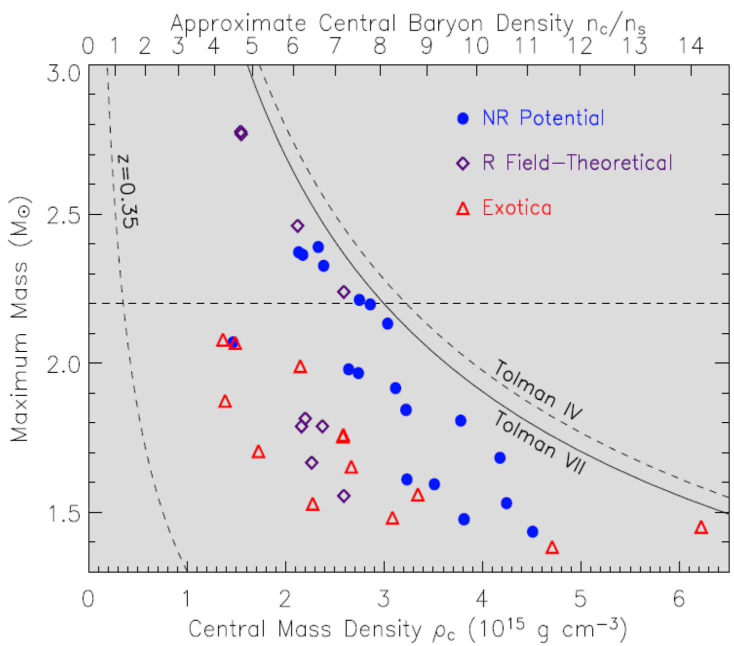
$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$$

$$\rho = \rho_c[1-(r/R)^2]$$

$$\rho_c = 15\beta c^2/(8\pi G R^2)$$

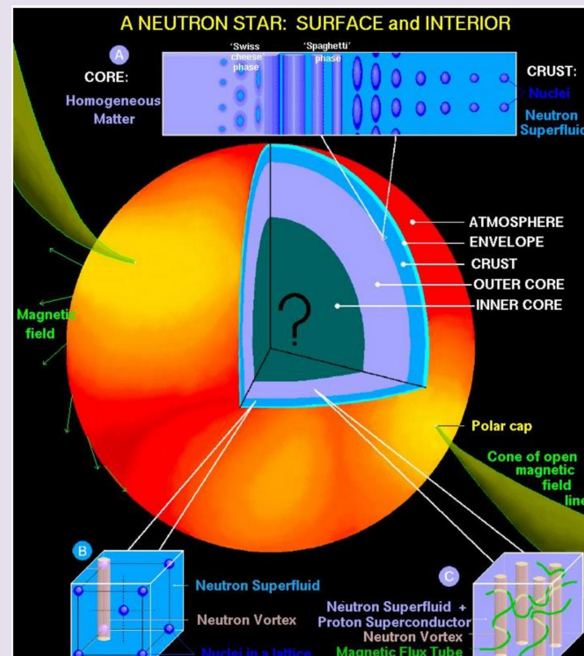
$$\beta = GM/Rc^2$$

Lattimer & Prakash, Astrop. J. 550:426-442, 2001; Raghoonundun, Hobill, PRD 92(2015) 124005.



Lattimer & Prakash, Phys.Rev.Lett. 94 (2005) 111101

FIG. 1: The maximum mass - central density relation predicted by causality coupled with the Tolman VII and Tolman IV analytic GR solutions are compared with structure integration results for a variety of EOSs.



Credit:
http://www.astroscu.unam.mx/neutrones/NS-picture/NStar/NStar-I.gif)

(Gravitational) radiation: pp-wave and plane gravitational wave exact solutions

- Perturbative plane waves solutions for GW are often used but the exact solutions have provided exact theoretical basis for their existence in Einstein's theory.
- In Brinkmann coordinates (*Math. Ann.* **18**: 119, 1925), the metric reads:

$$ds^2 = H(u,x,y)\,du^2 + 2\,du\,dv + dx^2 + dy^2$$

- *H(u,x,y)* is a smooth function. An important class of symmetric pp-waves is the plane waves exact solution given by:

$$H(u,x,y) = a(u)\,(x^2 - y^2) + 2\,b(u)\,xy + c(u)\,(x^2 + y^2)$$

- *a(u)* and *b(u)* describe the 2 polarization modes
- *c(u)=0* : the solution describes vacuum plane waves or plane gravitational waves (Baldwin, Jeffery, *Proc. Roy. Soc. Lond. A* 111 (757): 95, 1926)
- Using the exact solution, some recent work on pulsar timing arrays claiming that non-linear effects can become important (Harte, CQG 32 (2015) 17, 175017)

Exact solutions as cosmological models

- The everywhere-isotropic Friedmann-Lemaître-Robertson-Walker (FLRW) standard model of cosmology:

$$ds^2 = -c^2 \mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)\right]$$

- A perfect fluid source $T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu}$
- EFE give the Friedmann equations:

$$\left(\frac{\dot{a}}{a}\right)^2 =: H^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda c^2}{3}$$

- Over 300 exact solutions as anisotropic or inhomogeneous cosmological models

  "An exact solution to Einstein equations is termed cosmological if it can reproduce a FLRW metric when its arbitrary constants or functions assume certain values or limits."
  (Krasinski, Inhomogeneous Cosmological Models, Cambridge U.P., 1997)

- There are invariantly well-defined conditions for an FLRW metric

Exact solutions as cosmological models: examples

- Szekeres family spatially inhomogeneous models, dust source. (Szekeres, Commun. Math. Phys. 41, 55–64, 1975).

- Szekeres-Szafron family of inhomogeneous models, pressure added to the source but with some known limitations (Szafron, J. Math. Phys. 18, 1673, 1677).

- Stephani-Barnes family of inhomogeneous models, energy density and pressure in the source but limitations arise with a barotropic equation of state. (Stephani, Commun. Math. Phys. 4, 137, 1967; Barnes, GRG 4, 105, 1973).

- The Lemaître-Tolman-(Bondi) inhomogeneous spherically symmetric models, dust source. (Lemaître, Ann. Soc. Sci. Bruxelles 53, 51, 1933; Tolman, Proc. Nat. Acad. Sci. U.S. 20, 169-176, 1934; Bondi, Mon. Roy. Astr. Soc. 107, 410, 1947).

- The Oleson inhomogeneous models with perfect fluid source (Oleson, J. Math. Phys. 12, 666, 1971).

- Bianchi family of models, spatially homogeneous but anisotropic, perfect fluid source (e.g. Ellis and MacCallum, Commun. Math. Phys. 12, 108–141, 1969)

- Kantowski-Sachs family of models, , spatially homogeneous but anisotropic, perfect fluid source (Kantowski and Sachs, J. Math. Phys. 7, 443–446 1966.)

The Lemaitre-Tolman(-Bondi) (LT or LTB) Spherically symmetric inhomogeneous models

- Lemaître, Ann. Soc. Sci. Bruxelles 53, 51, 1933; Tolman, Proc. Nat. Acad. Sci. U.S. 20, 169-176, 1934; Bondi, Mon. Roy. Astr. Soc. 107, 410, 1947:

$$ds^2 = -\mathrm{d}t^2 + \frac{(R_{,r})^2}{1-k(r)}\mathrm{d}r^2 + R^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)$$

- EFE with a Λ give

$$\left(\frac{R_{,t}}{R}\right)^2 = \frac{2M}{R^3} - \frac{k}{R^2} + \frac{\Lambda}{3} \qquad \rho = \frac{2M_{,r}}{R_{,r}\,R^2}$$

where $R(t,r)$ is a generalized scale factor and $M(r)$ the active gravitational mass

- $t_B(r)$, the bang time

$$t - t_B(r) = \int_0^R \frac{\mathrm{d}\tilde{R}}{\sqrt{-k + \frac{2M}{\tilde{R}} + \frac{\Lambda}{3}\tilde{R}^2}}$$

- Extensively used for the topic of void-models for the cosmic acceleration problem

- Used for spherical collapse and exact modeling of structure formation (e.g. Hellaby, Krasinski, PRD, 73, 023518, 2006; Sussman, CQG, 30 (2013) 065016).

Szekeres Inhomogeneous cosmological models

- Originally derived by Szekeres (Comm. Math. Phys. 41, 55 (1975); PRD 12, 10 (1975)). The metric is

$$ds^2 = -\mathrm{d}t^2 + \mathrm{e}^{2\alpha}\mathrm{d}r^2 + \mathrm{e}^{2\beta}(\mathrm{d}p^2 + \mathrm{d}q^2)$$

where $\alpha = \alpha(t,r,p,q)$ and $\beta = \beta(t,r,p,q)$

- Two classes depending on $\beta_{,r} = 0$ or $\beta_{,r} \neq 0$. For example for $\beta_{,r} \neq 0$

$$\mathrm{e}^{\beta} = \Phi(t,r)\mathrm{e}^{\nu}(r,p,q),$$

$$\mathrm{e}^{\alpha} = h(r)\Phi(t,r)\beta_{,r} := h(r)(\Phi_{,r} + \Phi\nu_{,r}),$$

$$\mathrm{e}^{-\nu} = A(r)(p^2+q^2) + 2B_1(r)p + 2B_2(r)q + C(r)$$

$$g(r) := 4(AC - {B_1}^2 - {B_2}^2) = 1/h^2(r) + k(r)$$

$$\left(\frac{\Phi_{,t}}{\Phi}\right)^2 = -\frac{k}{\Phi^2} + \frac{2M}{\Phi^3} \qquad \rho = \frac{(2M\mathrm{e}^{3\nu})_{,r}}{\mathrm{e}^{2\beta}(\mathrm{e}^{\beta})_{,r}}$$

- Dust source
- No symmetries: no Killing vectors (Bonnor, Sulaiman, Tomimura, GRG, 8, 549, 1977)

Szekeres Inhomogeneous models as exact perturbations on a smooth background

- Reformulated by Goode & Wainwright (MNRAS 198, 83, 1982; PRD 26, 3315, 1982). The metric reads

$$ds^2 = -\mathrm{d}t^2 + S^2\left[e^{2\nu}(\mathrm{d}\tilde{x}^2 + \mathrm{d}\tilde{y}^2) + H^2W^2\mathrm{d}r^2\right]$$

where: $H(t, r, \tilde{x}, \tilde{y}) = A(r, \tilde{x}, \tilde{y}) - F(t, r)$

- The EFE give $\ddot{F} + 2\frac{\dot{a}}{a}\dot{F} - \frac{3M}{a^3}F = 0$ $\quad \left(\frac{\dot{a}}{a}\right)^2 = \frac{2M}{a^3} - \frac{k}{a^2}$

$$\rho = \frac{6MA}{a^3H} = \frac{6M}{a^3}\left(1 + \frac{F}{H}\right)$$

- time and space dependencies for each of the two classes are given in (Goode and Wainwright 1982a, 1982b).

- Shows the models as exact perturbations on the top of a smooth background allowing comparisons with perturbation theory and growth of structures (Goode and Wainwright 1982; Meures & Bruni, PRD 83:123519,2011; MI, Peel, PRD 85, 083502, 2012)

Szekeres Inhomogeneous models in LT-type coordinates

- In the a Lemaitre-Tolman type of coordinates (e.g. Hellaby, Krasinski, PRD 66:084011,2002)

$$ds^2 = -dt^2 + \frac{(R_{,r} - R\frac{E_{,r}}{E})^2}{\epsilon - k(r)} dr^2 + \frac{R^2}{E^2}(dp^2 + dq^2)$$

- where the metric functions satisfy the constraints

$$E(r,p,q) = \frac{(p^2+q^2)}{2S(r)} - \frac{P(r)}{S(r)}p - \frac{Q(r)}{S(r)}q + C(r)$$

$$C(r) = \frac{P^2(r)}{2S(r)} + \frac{Q^2(r)}{2S(r)} + \frac{S(r)}{2}\epsilon$$

- The EFE give:

$$(R_{,t}(t,r))^2 = \frac{2M(r)}{R(t,r)} - k(r)$$

$$8\pi\rho(t,r,p,q) = \frac{2(M_{,r}(r) - 3M(r)\frac{E_{,r}}{E})}{R^2(R_{,r} - R\frac{E_{,r}}{E})}$$

- Makes comparisons to LTB and FLRW geometries simpler

- hyperbolic spatial sections k(r)<0, parabolic k(r)=0, and elliptic k(r)>0
- ε=0,+1,or -1 determines how the 2-surfaces (p,q) of constant r foliate the spatial sections of constant t
- E(r,p,q) determines the mapping of p and q onto the 2-surface of constant t and r.

Szekeres Inhomogeneous models in LT-type coordinates

The time evolution in the 3 cases:

The hyperbolic case: $k(r) < 0$

$$R(t,r) = \frac{M(r)}{-k(r)}(\cosh\eta - 1)$$

$$t - t_B(r) = \sigma\frac{M(r)}{(-k(r))^{3/2}}(\sinh\eta - \eta)$$

The parabolic case: $k(r) = 0$

$$R(t,r) = M(r)\frac{\eta^2}{2}$$

$$t - t_B(r) = \sigma M(r)\frac{\eta^3}{6}$$

The elliptic case: $k(r) > 0$

$$R(t,r) = \frac{M(r)}{k(r)}(1 - \cos\eta)$$

$$t - t_B(r) = \sigma\frac{M(r)}{(k(r))^{3/2}}(\eta - \sin\eta)$$

The spatial functions are specified using large scale structure fittings or linked to various initial conditions (Bolejko PRD 73:123508,2006; MI, Peel, Troxel, PRL. 111, 251302, 2013)

The null geodesics equations in the Szekeres models

The affinely parameterized null geodesic equations, $k^{\alpha}{}_{;\beta}k^{\beta} = 0$, read (e.g. Nwankwo, MI, Thompson, JCAP 1105:028, 2011)

$$\dot{k}^t + \frac{R_{,tr} - R_{,t}\frac{E_{,r}}{E}}{\epsilon - k}\left(R_{,r} - R\frac{E_r}{E}\right)(k^r)^2 + \frac{RR_{,t}}{E^2}[(k^p)^2 + (k^q)^2] = 0$$

$$\dot{k}^r + 2\frac{R_{,tr} - R_{,t}\frac{E_{,r}}{E}}{R_{,r} - R\frac{E_{,r}}{E}}k^t k^r + \left(\frac{R_{,rr} - R_{,r}\frac{E_{,r}}{E} - R\frac{E_{,rr}}{E} + R(\frac{E_{,r}}{E})^2}{R_{,r} - R\frac{E_{,r}}{E}} + \frac{k_{,r}}{2(\epsilon - k)}\right)(k^r)^2$$
$$+2\frac{R}{E^2}\frac{E_{,r}E_{,p} - EE_{,pr}}{R_{,r} - R\frac{E_{,r}}{E}}k^r k^p + 2\frac{R}{E^2}\frac{E_{,r}E_{,q} - EE_{,qr}}{R_{,r} - R\frac{E_{,r}}{E}}k^r k^q - \frac{R}{E^2}\frac{\epsilon - k}{R_{,r} - R\frac{E_{,r}}{E}}[(k^p)^2 + (k^q)^2] = 0$$

$$\dot{k}^p + 2\frac{R_{,t}}{R}k^t k^p - \frac{1}{R}\frac{R_{,r} - R\frac{E_{,r}}{E}}{\epsilon - k}(E_{,r}E_{,p} - EE_{,pr})(k^r)^2 + 2\left(\frac{R_{,r}}{R} - \frac{E_{,r}}{E}\right)k^r k^p$$
$$-\frac{E_{,p}}{E}(k^p)^2 - 2\frac{E_{,q}}{E}k^p k^q + \frac{E_{,p}}{E}(k^q)^2 = 0$$

$$\dot{k}^q + 2\frac{R_{,t}}{R}k^t k^q - \frac{1}{R}\frac{R_{,r} - R\frac{E_{,r}}{E}}{\epsilon - k}(E_{,r}E_{,q} - EE_{,qr})(k^r)^2 + 2\left(\frac{R_{,r}}{R} - \frac{E_{,r}}{E}\right)k^r k^q$$
$$+\frac{E_{,q}}{E}(k^p)^2 - 2\frac{E_{,p}}{E}k^p k^q - \frac{E_{,q}}{E}(k^q)^2 = 0$$

where $k^{\alpha} = \frac{dx^{\alpha}}{ds}$, and $\dot{} = \frac{d}{ds}$. the LTB equations are obtained by setting ε=1 and $E_{,r} = 0$

Distances in Szekeres Inhomogeneous Cosmological models

The area-distance, luminosity distance and redshift are calculated from: (e.g. Nwankwo, MI, Thompson, JCAP 1105:028, 2011)

$$d\ln D_A = \theta \; ds = \frac{1}{2} k^{\alpha}{}_{;\alpha} \; ds. \qquad D_L = (1+z)^2 \, D_A.$$

$$\begin{aligned} d\ln D_A \;=\; & \Big[\frac{1}{2}\left(\frac{(F^2)_{,t}}{F^2}k^t + \frac{(F^2)_{,r}}{F^2}k^r + \frac{(F^2)_{,p}}{F^2}k^p + \frac{(F^2)_{,q}}{F^2}k^q\right) + \\ & \frac{1}{4}\left(\frac{H_{,t}}{H}k^t + \frac{H_{,r}}{H}k^r + \frac{H_{,p}}{H}k^p + \frac{H_{,q}}{H}k^q\right) + \frac{1}{2}\left(k^t_{,t} + k^r_{,r} + k^p_{,p} + k^q_{,q}\right)\Big] ds \end{aligned}$$

$$\frac{1}{1+z}\frac{dz}{ds} = \frac{1}{k^t}\frac{dk^t}{ds} = -\frac{1}{2\,k^t}\left[(H)_{,t}(k^r)^2 + \frac{(F^2)_{,t}}{F^2}((k^t)^2 - H(k^r)^2)\right]$$

- Where $H = \dfrac{(R_{,r} - R\frac{E_{,r}}{E})^2}{\epsilon - k}$ and $F = \dfrac{R}{E}$
- Alternatively, the Sachs equations can be used to get θ (e.g. Bolejko, Celerier PRD 82:103510,2010)

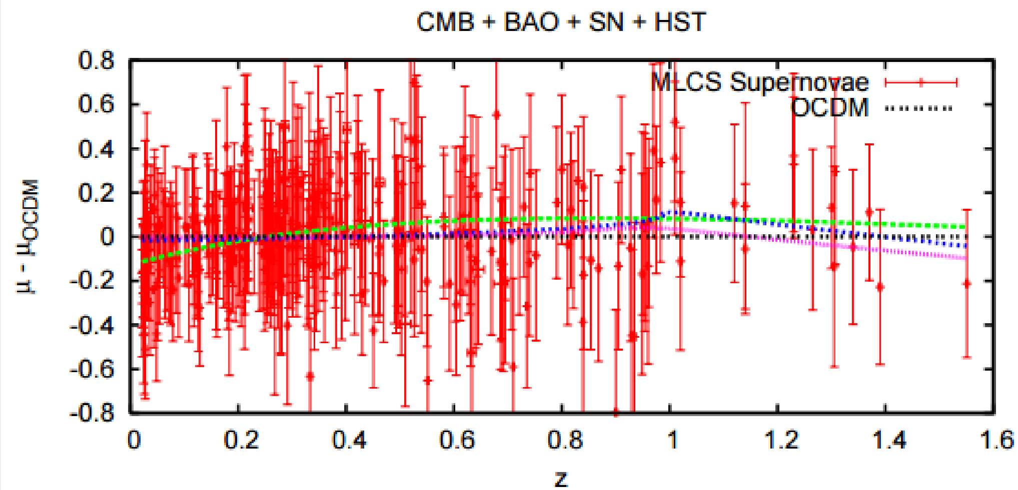
$$\frac{\mathrm{d}\theta}{\mathrm{d}s} + \theta^2 + |\sigma|^2 = -\frac{1}{2}R_{\alpha\beta}k^\alpha k^\beta,$$

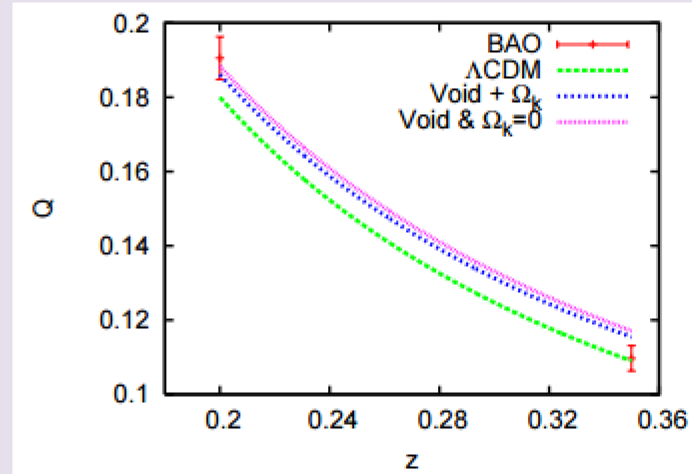
$$\frac{\mathrm{d}\sigma}{\mathrm{d}s} + 2\theta\sigma = -\frac{1}{2}R_{\alpha\beta\mu\nu}\epsilon^{*\alpha}k^\beta\epsilon^{*\mu}k^\nu$$

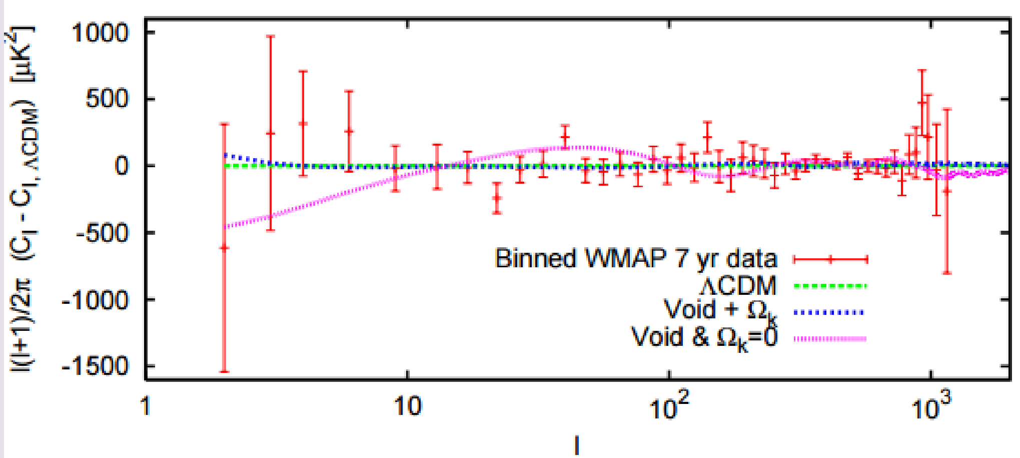
Application: cosmic acceleration

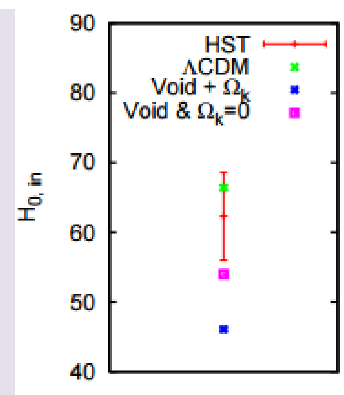
- Although less of a debate in last 2 years, the question of apparent acceleration due to observations in an inhomogeneous model attracted a lot of interest (e.g. reviews Enqvist, GRG 40, 451, 2008; Marra & Notari, CQG, 28, 164004, 2011)
- In inhomogeneous models, the Hubble expansion is a function of space and time so spatially uneven expansion can be modeled
- An observer in an underdense region expanding faster than the average can see an apparent accelerated expansion.
- The cosmological Copernican principle was dropped
- Kilo-parsec to Mega-parsec underdense region were found inconsistent with observations but a Giga-parsec underdense region model persisted (e.g. Biswas, Notari, W. Valkenburg, JCAP 1011, 030, 2010)

Observations in a Giga-parsec scale underdense region ("void")
(e.g. Biswas, Notari, Valkenburg, JCAP 1011, 030, 2010)









Model	CMB	BAO	SN	$HST_{62\pm6}$	total χ^2
ΛCDM	3371.1	3.1	239.5	0.4	**3614.1**
Profile A (Curved Void)	3377.4	4.0	238.9	4.1	3624.4
Profile B	3377.0	0.2	237.9	2.2	3617.3
Profile C	3376.9	0.7	237.7	1.9	3617.2
Profile D	3377.5	3.6	233.7	1.3	**3616.1**
Profile E	3380.2	3.3	241.4	0.8	3625.7

Challenges to apparent acceleration

- Even if the Copernican principle is dismissed, large-void models have been criticized in the literature.

- Some papers found LTB large voids in tension with data sets: e.g. (Moss, Zibin, Scott, PRD 83:103515,2011) found he models predict a local $h_0 \approx 0.45 \pm 0.02$ and also in tension with BAO (in disagreement with (Biswas, Notari, W. Valkenburg, JCAP 1011, 030, 2010)

- (Zibin and Moss, CQG 28:164005,2011) found that a large class of void models are ruled out as they predict a much larger kinetic-Sunayev-Zeldovich power than observed, but mention some caveats

- Using the Szekeres models, (MI, Peel, Troxel, PRL 111, 251302, 2013) found that the models that fit well supernova data fail to reproduce the suppression of growth of large scale structure, but with some caveats.

Application: exact perturbations and growth of structures

- Using LTB (e.g. Sussman, PRD 79:025009,2009); Using Szekeres (Kasai, PRD 47, 3214, 1993, Bolejko PRD 73:123508,2006; Meures & Bruni PRD 83:123519,2011; Peel, MI, Troxel PRD 86,123508,2012; Sussman, Gaspar, PRD 92, 083533, 2015)

- For example, in Szekeres $\rho(t,r,\tilde{x},\tilde{y}) = \rho_q(t,r)[1+\hat{\delta}(t,r,\tilde{x},\tilde{y})]$

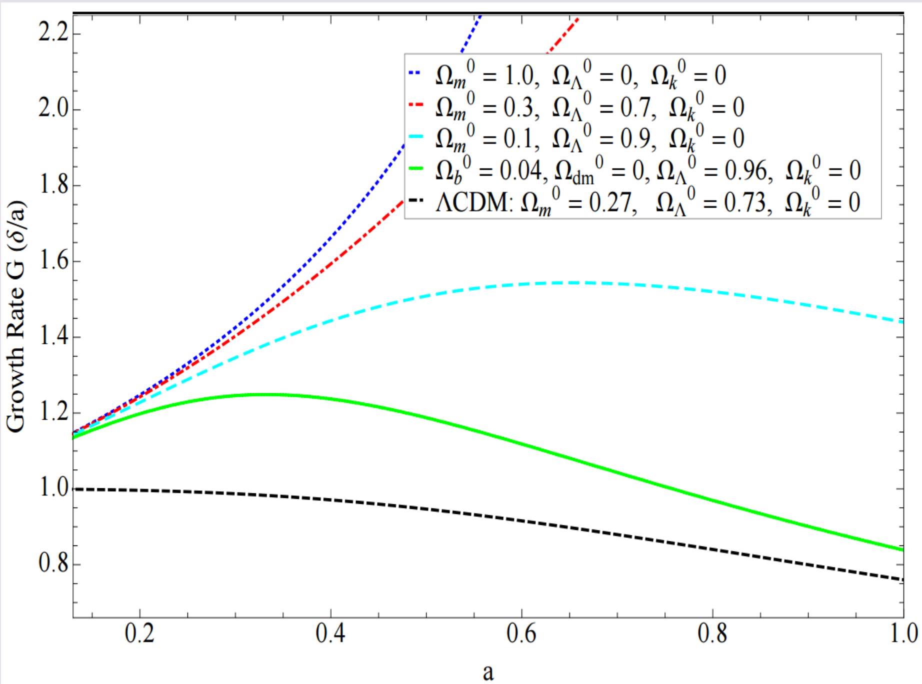
$$\rho(t,r,\tilde{x},\tilde{y}) = \rho_q(t,r)[1+\hat{\delta}(t,r,\tilde{x},\tilde{y})], \qquad \rho_q(t,r) = \frac{6M(r)}{a^3(t,r)}$$

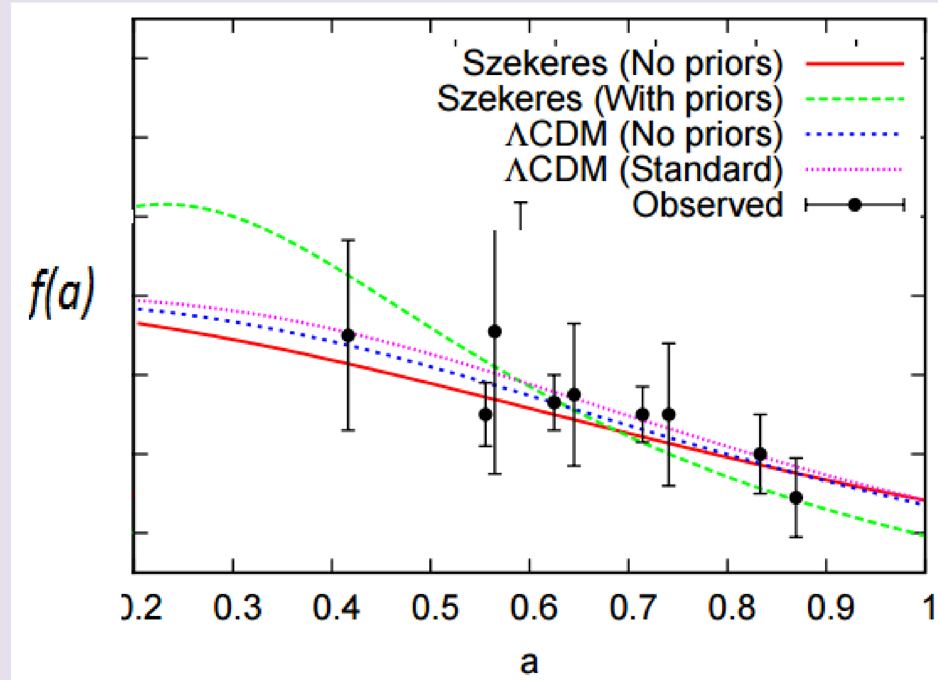
$$\rho = \frac{6MA}{a^3H} = \frac{6M}{a^3}\left(1+\frac{F}{H}\right) \qquad \hat{\delta}(t,r,\tilde{x},\tilde{y}) = \frac{F(t,r)}{H(t,r,\tilde{x},\tilde{y})}$$

$$\hat{\delta}'' + \left(\frac{4+2\Omega_\Lambda-\Omega_m}{2a}\right)\hat{\delta}' - \frac{3}{2}\frac{\Omega_m}{a^2}\hat{\delta} - \frac{2}{1+\hat{\delta}}\hat{\delta}'^2 - \frac{3}{2}\frac{\Omega_m}{a^2}\hat{\delta}^2 = 0$$

- This is an exact expression. First 3 terms are similar to those of an FLRW model. δ is not restricted to be smaller than 1.

Application: Exact perturbations and growth of structures





(Peel, MI, Troxel PRD 86,123508,2012)

- The growth is stronger in Szekeres
- Can fit observations but with different cosmological parameters than LCDM

	$L\alpha$?	Without priors Ω_m^0	Ω_Λ^0	Ω_k^0	(χ^2)	With priors Ω_m^0	Ω_Λ^0	Ω_k^0	(χ^2)
Szekeres	No	0.12	0.59	0.29	(0.22)	0.05	0.98	-0.03	(0.62)
	Yes	0.11	0.69	0.20	(0.39)	0.05	0.98	-0.03	(0.63)
ΛCDM	No	0.29	0.56	0.15	(0.21)	0.27	0.73	0.00	(0.32)
	Yes	0.26	0.69	0.05	(0.39)	0.27	0.73	0.00	(0.43)

$$0.039 \leq \Omega_b^0 \leq 0.049 \text{ from BBN (with } \Omega_{dm}^0 = 0)$$

Application: The effect of inhomogeneities on cosmological parameter estimation using exact solutions

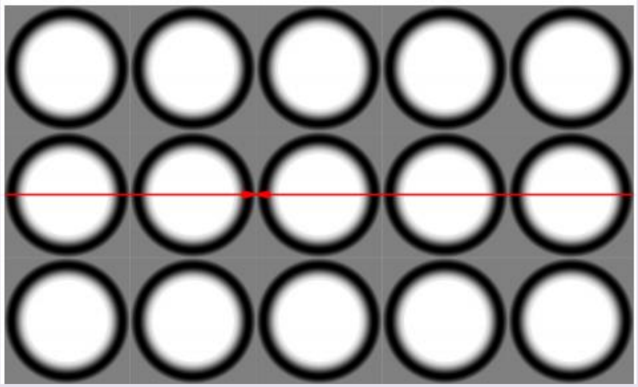
- Cosmological constructions using exact matching of spacetimes where the observer is not at a privileged location

- By the Darmois matching spacetime conditions, these are still exact solutions to EFE

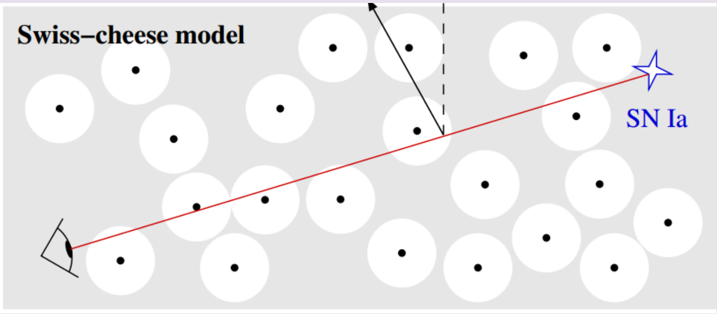
- Studies find residual effects on redshift and distance observables (e.g. reviews by Ellis, CQG, 28, 164001 , 2011; Marra & Notari, CQG, 28, 164004, 2011)

- A common finding is that observables are affected leading to small but non negligible effect on cosmological parameters.

- Are these effects at the level of systematic effects in cosmological data?



Marra, Kolb, Matarrese, PRD 77, 023003 , 2008



Fleury, Dupuy, Uzan, PRD 87, 123526, 2013

The averaging problem in relativity and cosmology

- Following the tradition in books and talks on exact solutions in cosmology: a few remarks on the averaging problem.

- The problem comes from the fact that spatial averaging and applying the Einstein's field equations are two operations that do not commute due to the non-linear nature of General Relativity. e.g. (Ellis, 1983; Ellis, CQG, 28, 164001 Zalaletdinov, gr-qc/0701116, 2007)

$$\left\langle \left(R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R \right) [g_{\rho\sigma}] \right\rangle \neq \left(\langle R_{\alpha\beta} \rangle - \frac{1}{2} \langle g_{\alpha\beta} R \rangle \right) [\langle g_{\rho\sigma} \rangle]$$

- Applying this to the lumpy universe gives "macroscopic" Einstein equations and Friedmann equations with extra terms, often called backreaction terms

- Several formalisms developed : e.g. scalar averaging by Buchert, GRG 32, 105, 2000; a covariant tensor approach (macroscopic gravity) by Zalaletdinov, GRG 24, 1015, 1992.

- Coley, Pelavas, Zalaletdinov, PRL 95 (2005) 151102 obtained for a flat FLRW macroscopic metric

$$\left(\frac{\dot{a}}{a} \right)^2 = \frac{\kappa\rho}{3} + \frac{\kappa\beta}{3a^2(t)}, \; 2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 = -\kappa p + \frac{\kappa\beta}{3a^2}.$$

The averaging problem in relativity and cosmology

- Macroscopic gravity compared to Cosmological distances and growth rate data: e.g. Clarkson et al. PRD 85, 043506, 2012; Wijenayake, MI, PRD 91, 063534, 2015 obtained:

  $$-0.03 \leq \Omega_{\mathcal{A}} \leq 0.05$$

  where Ω_A represents the backreaction parameter

- Comparison to full CMB and other data sets is needed (Wijeneyake, MI, in preparation, 2016)

- While the backreaction terms have been found small, their effect on cosmological parameter constraints in precision cosmology is still an open question.
  (e.g. reviews: Rasanen, CQG, 28, 164008, 2011; Clarkson & Umeh, CQG, 28, 164010, 2011)

Concluding Remarks

- A large number of exact solutions are available
- Some still need physical analyses and interpretations
- They can provide mathematical and physical insights
- Some solutions have been influential in the development of relativity, cosmology and astrophysics
- They can provide useful comparisons or validations of approximations and numerical work
- They can constitute a starting point for simulations
- Interactive online databases available now on the internet and may enhance their accessibility and analysis
- Being exact, they are in general hard to adapt for more specific modeling purposes
- They represent part of the legacy of the Einstein's theory of Gravity.