Chapter 1
Fitting a drift-diffusion Item Response Theory model to complex cognition response times

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Abstract Drift Diffusion Models (DDM) have been widely successful in modeling fast decision response times. DDM describes the underlying (cognitive) decision process as a function of a diffusion process drifting toward a decision threshold. A few studies have shown that introducing within-trial variability in DDM parameters or describing DDM parameters as a function of item properties improves the DDM model fit for response times of the Complex Decision Task (CDT) as well. One such extension of DDM is the Item Response theory-based Q-diffusion model (QDM). QDM has been successful in modeling response times of CDT such as chess ability assessment. The current study further examined whether QDM can fit response times corresponding to certain problem-solving tasks. First, the drift rate parameter of standard DDM was extended to approximate the within-trial variability in the reasoning process as discussed in existing meta-reasoning studies that examine such within-trial dynamics for problem-solving tasks. Then, the response times were simulated using the standard DDM and the mentioned extension of the standard DDM. Then, the goodness-of-fit of QDM was examined using a Bayesian model fit method - Posterior Predictive Check (PPC). PPC analysis revealed that the fitted QDM was able to effectively describe the simulated response time mean. However, the fitted QDM was not able to describe the simulated response time variance.

1.1 Introduction

The Drift Diffusion Model (DDM) is a process model of decision tasks and has been widely successful in modeling the response accuracy, response times, and final confidence judgments (subjective judgment in the likelihood of being accurate) of fast two-choice decision tasks (see Ratcliff et al., 2016, for a review). Also,

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several extensions of DDM have been proposed to model complex decision tasks such as risky gambling task (Diederich and Trueblood, 2018), and ability-based assessments, e.g., chess ability (van der Maas et al., 2011). Also, DDMs have been utilized to study the underlying mechanisms of several psychological conditions such as attention deficit (Feldman and Huang-Pollock, 2021) and emotion regulation (Warren et al., 2020). Overall, DDMs have been found helpful in examining the underlying cognitive mechanisms of a decision-making process.

A complex decision task may require an individual to reason through multiple potential solutions to select a single response with a higher probability of being accurate. Reasoning can be understood as a multi-step cognitive search process over a mental space of solution states associated with a problem at hand. While a reasoning process is being executed, the meta-reasoning processes estimate the likelihood of success of the current reasoning process through multiple heuristic cues such as familiarity (Ackerman, 2019). Meta-reasoning studies typically examine the growth patterns of subjective confidence levels by requiring participants to elicit their intermediate confidence judgment on the likelihood of being accurate in the current state of their response. (see Ackerman and Thompson, 2017, for a review).

These meta-reasoning studies suggest that the evolution pattern of the reasoning process can evolve in both linear and non-linear (e.g., piecewise, exponential, logarithmic) growth patterns (Ackerman, 2014). Also, these evolution patterns depend on the participant's initial state of confidence level and item characteristics. DDMs have been widely utilized to generate such linear and non-linear growth patterns, e.g., the Wiener process and the Ornstein-Uhlenbeck (OU) process respectively. In DDMs, the probability of transitioning from the current state to the next state is defined using a drift rate parameter. The drift rate parameter influences the evolution pattern and direction of the state transition. A constant drift rate within a decision process typically would lead to a linear evolution of the decision process, whereas the variable drift rate (variability introduced based on current state or time) typically would lead to a non-linear evolution of the decision process (Diederich and Busemeyer, 2003).

Typically, DDMs utilized to model complex decision tasks consider drift rate to be variable within a trial (e.g., Diederich and Trueblood, 2018). Expressing the constant drift rate in terms of participant and item properties has also been utilized to model complex decision response times. This model is known as the Q-diffusion model (QDM) and was proposed by van der Maas et al. (2011). QDM has been successful in modeling response accuracy and response time of complex decision tasks such as the chess ability test (van der Maas et al., 2011) and the English spelling test (Rijn and Ali, 2017).

The current study further examined whether QDM can fit response time distributions that may result due to the within-trial evolution patterns of the reasoning process as discussed in meta-reasoning studies. First, a constant drift rate parameter was used to describe the fast and slow linear growth decision processes. Then, to approximate exponential and piecewise growth in the decision process (as discussed in meta-reasoning studies, e.g., Ackerman (2014)), state-based exponential and piecewise functions were proposed to introduce variability in the drift rate pa-

rameter. Then, these diffusion processes were utilized to simulate response accuracy and response time distributions. Then, QDM was fitted to each of these simulated response accuracy and response time distributions using Bayesian inference methods. Then, the goodness of fit of the fitted QDM was examined using Bayesian model fit evaluation methods.

1.2 Describing Diffusion process as a Markov Random walk process

A diffusion process, such as the Wiener process, can be understood as a Markov random walk in a space of a finite number of states (Diederich and Busemeyer, 2003). The diffusion random walk starts from an initial state and then transitions to the next state based on a transition matrix. Diederich and Busemeyer (2003) discussed that a birth-death transition matrix may be utilized to create the required transition matrix for the diffusion process. A birth-death transition matrix describes the probability of the random walk transitioning from the current to the next or the previous state, or remaining in the current state. Also, Diederich and Busemeyer (2003) proposed a method to create the birth-death transition matrix using the drift rate parameter and a pre-defined unit of time-step (see Eq. 15 of Diederich and Busemeyer, 2003). The birth-death transition matrix used in the current study, Eq. 1.1, is a simplified version of the birth-death transition matrix proposed by Diederich and Busemeyer (2003).

$$
p_{s,j} = \begin{cases} \frac{1}{2\alpha} \left(1 - v(s)\sqrt{\tau}\right) & \text{if } j - s = -1 \\ \frac{1}{2\alpha} \left(1 + v(s)\sqrt{\tau}\right) & \text{if } j - s = +1 \\ 1 - \frac{1}{\alpha} & \text{if } j = s \\ 0 & \text{otherwise} \end{cases} \tag{1.1}
$$

Here, $p_{s,j}$ describes the probability of the random walk transitioning from the sth state to the jth state, $v(s)$ is the drift rate parameter which can be defined either as a constant or as a function of the current state s, and τ is the pre-defined unit of time-step. The α parameter is also pre-defined and is chosen to be larger than 1 so that the discrete-time diffusion random walk, Eq. 1.1, closely resembles the continuous-time diffusion process. Also, the transition probability $p_{0,1} = p_{S,S-1} = 0$ to ensure that the diffusion process stops when either 0 or S decision threshold boundary is reached.

The Eq. 1.1 describes a Wiener process, if the drift rate parameter $v(s)$ is defined as constant across states s. Also, describing $v(s) = \delta - \gamma \cdot s$ represents the Ornstein-Uhlenbeck (OU) process (Diederich and Busemeyer, 2003). Here, δ describes the rate and direction of the transition and $\gamma \cdot s$ describes the decay in the rate of transition.

The random walk stops when it reaches the upper or the lower decision threshold. The response time can then be calculated by multiplying the number of steps taken to reach the decision boundary by τ. Also, typically, the response is considered accurate

if the random walk reaches the upper boundary. Hence, such a random-walk diffusion process can be utilized as a data-generating process to generate response accuracy and response time distribution for a desired decision process.

1.3 Simulating Response Distributions using DDM random walk

In the current study, the reasoning response accuracy and response times were simulated using the random walk diffusion process utilizing the transition matrix described in Eq. 1.1. The below section describes the drift rate configurations used to approximate the reasoning processes. Also, the below section describes how the drift rate varied across the simulated population. Also, the below section describes the random walk method used to simulate the response data.

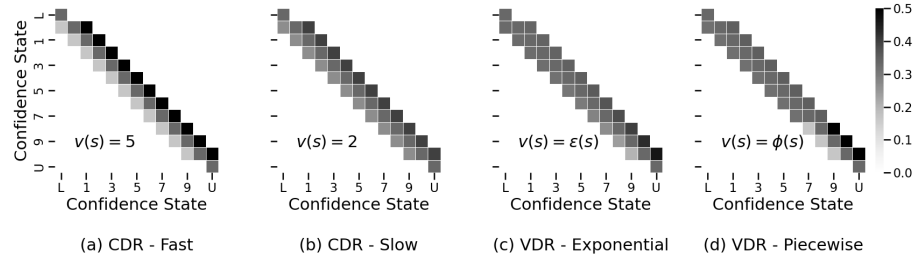### 1.3.1 Drift Rate Configurations

Fig. 1.1 The impact of four proposed configurations of positive drift rates, $v(s)$, on a 13-state transition matrix expressing 0-10% confidence states. The additional 2 states, L and U describe the Lower and Upper decision boundary. Here, CDR means Constant Drift Rate and VDR means Variable Drift Rate. A lighter (or darker) colored box depicts a lower (or higher) probability of transitioning from the current state (s) shown in the y-axis to the next states shown in the x-axis. The exponential $\epsilon(s)$ and piecewise $\phi(s)$ functions are described in Eq. 1.2 and Eq. 1.3, respectively.

A constant drift rate $v(s) = 5$ and $v(s) = 2$ across all states was used to calculate the birth-death transition matrix describing fast and slow diffusion processes (see Fig 1.1a and 1.1b respectively). The proposed state-based exponential function $\epsilon(s)$ used to introduce variability in the drift rate $v(s)$ was: (see Fig 1.1c for the resulting birth-death transition matrix).

$$\epsilon(s) = \frac{e^{y(s)}}{b}; \quad y(s) = v_s + s \times \rho; \quad \rho = \frac{(v_e - v_s)}{(S-1)}; \tag{1.2}$$

Here, the start v_s and end v_e drift rates were predefined to be 0.04 and 4 respectively. The b parameter describes a constant used to reduce the magnitude of $\epsilon(s)$ so that the drift rates do not get too large, s is the current state, and $S = 103$ is the total number of states.

The proposed state-based piecewise function $\phi(s)$ used to introduce variability in the drift rate $v(s)$ was (see Fig 1.1d for the resulting birth-death transition matrix)

$$\phi(s) = \begin{cases} 0.5 & \text{if } 0 < s < 75 \\ 5 & \text{if } 76 < s < 103 \end{cases} \tag{1.3}$$

The birth-death transition matrix described the diffusion process in terms of confidence levels 0-100% (see Fig. 1.1 for a 13-state transition matrix); (also see Appendix C of Pleskac and Busemeyer, 2010, for a related discussion). The total number of states was kept at 103, where the additional 2 states represented the upper and lower decision threshold boundaries.

To introduce variability in drift rates among the participant population, a small noise, sampled from Normal$(0, 0.01)$ distribution, was added to the constant and variable drift rates. Also, given a positive (or negative) drift rate describes a higher probability of getting correct (or incorrect) responses, half of the participant population was assigned positive drift rates and the other half were assigned negative drift rates. Then, the transition matrix required for the random walk was calculated using the Eq. 1.1.

1.3.2 Random Walk

The response accuracy and response time distributions were simulated for 500 participants and 20 tasks. The initial state of the random walk was fixed at the center of the 103 confidence states, i.e., 51st confidence state. This reflects the idea that typically in reasoning tasks, participants may be unbiased towards any possible response options for a given task. Then, based on the transition matrix, the probability of transitioning to the neighboring states was identified. Then, the multinomial distribution function was used over these identified transition probabilities to draw the next transition state. This process was repeated until the upper or the lower decision boundary was reached (see Fig. 1.2). The response time was calculated by multiplying the number of steps taken to reach the upper (or lower) boundary and the smallest unit of time ($\tau = 0.01$) assumed for each step. Also, assuming the upper (and lower) boundary represents the correct (and incorrect) response, response accuracy was calculated based on the boundary threshold that the random walk reached (see Fig 1.2).

1.4 The Q-diffusion Model (QDM)

The QDM, proposed by van der Maas et al. (2011), is an Item Response Theory-based extension of the DDM that has no within-trial variability in the parameters. In QDM, the probability of reaching an upper or a lower decision boundary in t time is modeled as, $\log(t) \sim Normal(\mu_t, \sigma_t^2)$, where

$$
\begin{aligned}
\mu_t &= \log(E_t) - \frac{1}{2}\left[1 + \frac{V_t}{E_t}\right] \quad \sigma_t^2 = \log\left[1 + \frac{V_t}{E_t}\right] \\
E_t &= \left(\frac{a}{2v}\right)\frac{1 - e^{-av}}{1 + e^{-av}} + t_e \qquad V_t = \left(\frac{a}{2v^3}\right)\frac{-2h^{e(-av)} - y^{e(-2av)} + 1}{(e^{-av} + 1)^2}
\end{aligned}
\tag{1.4}
$$

Here, E_t and V_t describe the expected value, and the variance of the response time, respectively. Also, the drift rate ($v = \frac{v^p}{v^i}$) and the boundary separation ($a = \frac{a^p}{a^i}$) parameters are factored into a participant (^{P}) and an item-specific (^{i}) parameter. The parameter v^i represents item difficulty, a^i represents item time pressure, v^p represents person-specific drift rate, and a^p represents person-specific boundary separation. QDM also contains a person-specific non-decision time parameter, t_e, that describes the time required to register a response once a decision has been made. Also, in QDM, the probability of a response being correct or incorrect is modeled as:

$$
P(x = 1|a, v) = P(x = 0|a, v) = \frac{1}{1 + e^{-av}}
\tag{1.5}
$$

1.5 Bayesian modeling of the Q-diffusion model

A Bayesian model is typically represented as a joint probability model, also known as parameter posterior distribution, of the likelihood function (e.g., Eq. 1.4 and Eq. 1.5), and a plausible prior distribution of the model parameters. The current study utilized the Bayesian formulation of QDM proposed by van der Maas et al. (2011). Also, the Bayesian QDM was implemented using the Python software package - PyMC v5.8.2 (Salvatier et al., 2015).

1.5.1 Bayesian Inference

To perform Bayesian inference, samples were drawn from the parameter posterior distribution using the No-U-Turn Sampler (Hoffman and Gelman, 2011), Markov Chain Monte Carlo (MCMC) method, implemented in Python software package - PyMC v5.8.2. Four MCMC sampling chains were initialized from different starting

points to sample from the posterior distribution. The $\hat{R}$ metric estimated MCMC
sample convergence. The $\hat{R}$ metric checks convergence by comparing the estimated
variance of the parameter estimates within a particular MCMC sampling chain with
the estimated variance across chains (Gelman et al., 2013). Overall, 5,500 samples
were drawn from the posterior distribution, out of which an initial 1,000 samples
were considered warm-up samples and were discarded from further analysis. These
parameter samples were further utilized to perform a model fit evaluation of the
fitted QDM.

1.5.2 Bayesian Model Fit Evaluation

The posterior Predictive Distribution (PrD) was utilized to perform the Bayesian
model fit evaluation. Posterior PrD is a marginal distribution of the unobserved
response data, marginalized over the posterior distribution of parameter θ (Gelman
et al., 2013). The current study described the posterior PrD, $P(\tilde{t}|t)$, of the unobserved
response time $\tilde{t}$ as:

$$P(\tilde{t}|t) = \int_{\Theta} P(\tilde{t}|\theta)P(\theta|t)d\theta \tag{1.6}$$

The predictive response dataset was sampled (predicted) from $P(\tilde{t}|t)$ for each of the
parameter samples drawn from the parameter posterior distribution $P(\theta|t)$. Then,
the model fit analysis was performed by visually comparing the expected $P(\tilde{t}|t)$
response dataset and the observed response dataset used to fit QDM. The expected
$P(\tilde{t}|t)$ response dataset was estimated by averaging the $P(\tilde{t}|t)$ response samples over
the $P(\theta|t)$ parameter samples.

1.6 Results

1.6.1 Simulated Dataset

The drift rate parameter influences the evolution pattern of the diffusion random
walk (see Fig. 1.2). For example, Fig. 1.2b shows that a lower drift rate, $v(s) = 2$,
introduced more uncertainty in the random walk patterns, resulting in more number
of steps required to reach a decision boundary as compared to Fig. 1.2a having a
drift rate $v(s) = 5$. Also, as can be seen in Fig. 1.2, varying the drift rate across states
also affected the diffusion random walk trajectory. The diffusion random walk with
variable drift rates (Fig. 1.2c and Fig. 1.2d) followed a different evolution pattern
than the constant drift rate diffusion random walk (Fig. 1.2a and Fig. 1.2b). Also,
the resulting response time density differed across the four drift rate configurations
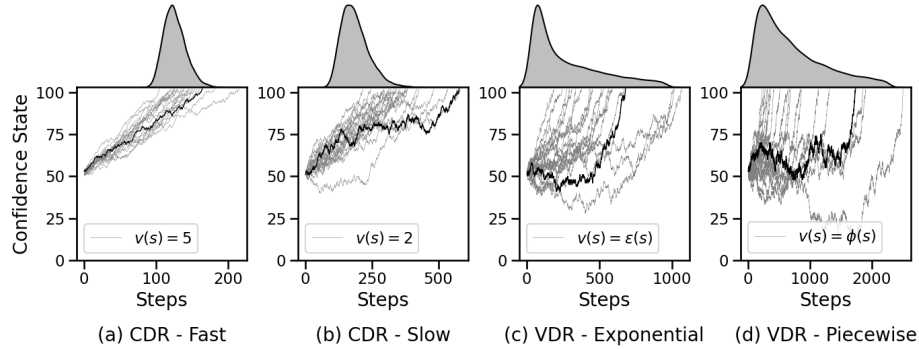(as depicted in Fig. 1.2).

Fig. 1.2 For each of the four configurations of positive drift rate parameter, $v(s)$, this figure shows the trajectories of the diffusion random walks performed over a 103 confidence level state space (shown in y-axis). Here, CDR means Constant Drift Rate and VDR means Variable Drift Rate. The exponential $\epsilon(s)$ and piecewise $\phi(s)$ functions are described in Eq. 1.2 and Eq. 1.3, respectively. The x-axis shows the number of steps taken by the random walk to reach a decision boundary. The density plot on top of each figure shows the density of response time generated by the random walks.

Expressing the drift rate as a piecewise function of the current state resulted in a random walk trajectory that initially stayed in the states surrounding the initial starting state. Then after reaching a slightly higher state, the random walk proceeded rapidly toward the decision boundary (see Fig. 1.2c). When expressing the drift rate as an exponential function of the current state, the random walk trajectory displayed a curved growth pattern that gradually increased its slope as the random walk reached higher states. The trajectory growth patterns were found to be similar for positive and negative drift rates (due to limited space, Fig. 1.2, only shows the trajectory for positive drift rates).

1.6.2 Model Fit

The model converged for all four drift rate configurations. The values of $\hat{R}$ for the four drift rate configurations were: 1) Fast: 1.004 ± 0.009, 2) Slow: 1.000 ± 0.002, 3) Exponential: 1.000 ± 0.001, and 4) Piecewise: 1.001 ± 0.004. The posterior predictive response times sampled from the fitted QDM model were in the log scale. This is because QDM describes the response time in the log scale (see Eq. 1.4). Hence, the model fit was examined by converting the simulated response times into the log scale as well. Fig. 1.3 compared the mean and standard deviation of the simulated response time datasets and the expected posterior predictive response time datasets. As can be seen, for all four drift rate configurations, the means of both simulated and expected posterior predictive response time datasets were similar. However, the standard deviations had lesser similarity for all four drift rate configurations.
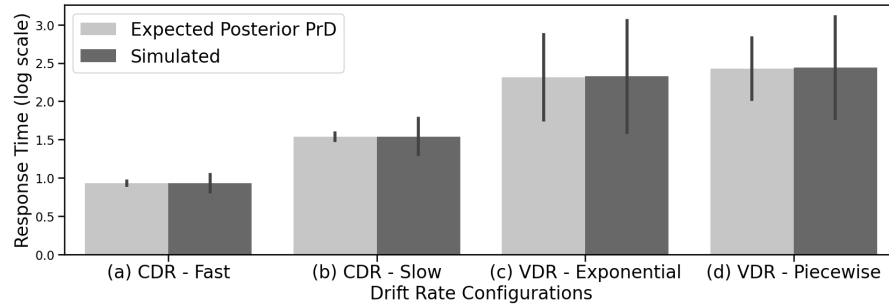
Fig. 1.3 For each of the four drift rate configurations, this figure compares the mean and standard deviation of the simulated response times and the expected posterior predictive response times, in log scale. Here, CDR means Constant Drift Rate and VDR means Variable Drift Rate.

1.7 Conclusion

Existing studies have shown that the item response theory-based extension of the diffusion model, Q-diffusion model (QDM), can be fitted to complex decision tasks such as Chess Ability assessment (van der Maas et al., 2011) and Spelling Test (Rijn and Ali, 2017). The current study further examined whether QDM can be fitted to problem-solving tasks as well. The current study found that the unobserved response times predicted from the fitted QDM were representative of the overall mean of the simulated problem-solving response times for all four simulation configurations. However, the current study also found that QDM was less effective in describing the overall variance in the simulated response times. The overall variance in a response time distribution is a result of between-trial and between-participant variations in response time. Hence, QDM can be further extended to model such between-trial and between-participant variations to better fit the overall variance in the response time distribution (see Kang et al., 2022, for a similar extension of item response theory diffusion model).

Also, the current study introduced state-based variability in the drift rate parameter of the standard diffusion random walk model. The drift rate was manipulated to approximate certain growth patterns of intermediate confidence judgments (ICJ) typically observed in the meta-reasoning studies administering problem-solving tasks (Ackerman, 2014; Metcalfe and Wiebe, 1987). The simulation results suggested that within-trial variability in the drift rate parameter impacts the tails of response time distributions differently than a constant within-trial drift rate parameter (see Fig. 1.2). Also, different within-trial drift rate configurations impact the tails of response time distributions differently (see Fig. 1.2c and 1.2d). Hence, the results indicated that a diffusion model (or item response theory-based extension of the diffusion model) designed to model complex cognition response times, may benefit by modeling the within-trial variations in the drift rate parameter (see Diederich and Trueblood, 2018, for a similar extension of the standard diffusion model).

Also, the current study only examined situations where ICJ grows (on average) within a task. However, some studies have shown that ICJ may either remain constant or may first deteriorate and then improve as time elapses during a complex cognitive task (Vernon and Usher, 2003). Also, in the current study, QDM was fitted to response time datasets where each task had the same characteristics within the dataset. However, in the ability-based educational assessments typically administering complex cognitive tasks, there would be a higher degree of variations in the administered task characteristics within the assessment. Hence, to further examine the applicability of QDM for such ability assessments, studies may be conducted using a response dataset simulated by combining item responses drawn from different diffusion random walk processes having different drift rate configurations, each approximating a different ICJ growth pattern.

1.8 Acknowledgment

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